MANTEL'S THEOREM

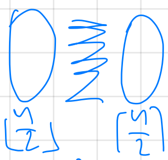
DEF: TURÁN'S NUMBER OF GRAPH H IS THE MAXIMUM OF EDGES AN H-FREE GRAPH ON n VERTICES CAN HAVE

$$ex(n, H) := \max_{\substack{G\text{-}n\text{VTX} \\ H\text{-FREE}}} e(G)$$

Q WHAT IS $ex(n, H) = ?$

THEOREM MANTEL (1907) $\quad ex(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor$

PROOF



NO K_3 & $ex(n, K_3) \geq e(G) = \left\lfloor \frac{n^2}{4} \right\rfloor$

$\left\lfloor \frac{n}{2} \right\rfloor \quad \left\lceil \frac{n}{2} \right\rceil$

G

U.B. LET G BE A K_3-FREE GRAPH ON n VTXS

$\forall e = x, y$

$\deg(x) + \deg(y) \leq n$ ← OTHERWISE THERE IS A TRIANGLE IF THEY HAVE A COMMON NEIGHBOUR

MAKES Δ IF z ADJACENT TO BOTH x & y

$$\sum_{x \in V} \deg^2(x) = \sum_{\substack{e = x, y \\ e \in E(G)}} \deg(x) + \deg(y) \leq \sum_{e(G)} n \leq |E(G)| \cdot n$$

HOW MANY TIMES IS $\deg(x)$ COUNTED ? FOR EACH EDGE INCIDENT WITH x IT IS COUNTED AS $\deg(x)$ SO TOTAL $\deg^2(x)$

$$\sum_{x \in V} \deg^2(x) \geq \frac{\left(\sum \deg(x)\right)^2}{n} = \frac{(2|E(G)|)^2}{n}$$

CS: $\left(\sum m_i n_i\right)^2 \leq \left(\sum m_i^2\right) \cdot \left(\sum n_i^2\right)$ ← USE IT AS $m_i = \deg(x)$, $n_i = \frac{1}{n}$

$$|E(G)| \cdot n \geq \frac{4|E(G)|^2}{n} \Rightarrow |E(G)| \leq \frac{n^2}{4}$$

STABILITY (NEW TOPIC)

BIPARTITE GRAPH IS NOT THE SAME AS BIPARTITE.

Q: HOW DIFFERENT ARE Δ-FREE AND BIPARTITE GRAPH

THEOREM FÜREDI 2012 K_3-FREE

LET G BE A TRIANGLE-FREE GRAPH ON n VERTICES WITH

$$e(G) = \left\lfloor \frac{n^2}{4} \right\rfloor - t \quad \text{FOR SOME } t \geq 0$$

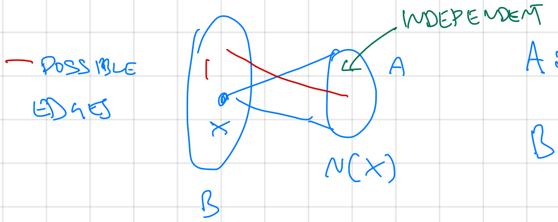
THEN THERE EXISTS A BIPARTITE SUBGRAPH $H \subseteq G$ S.T.

$$e(H) \geq e(G) - t$$

YOU CAN MAKE ↑ BIPARTITE BY REMOVING $\leq t$ EDGES

PROOF:

LET G BE A K_3-FREE GRAPH ON n VERTICES. LET x BE A VERTEX OF MAXIMUM DEGREE.



$A := N(x)$ $\qquad$ $|A| + |B| = n$

$B := V \setminus N(x)$

$|A| = \deg(x) = \Delta(G)$

$$\sum_{y \in B} \deg(y) = e(G) + e(G[B])$$

ALSO $\displaystyle\sum_{y \in B} \deg(y) \leq \sum_{y \in B} |A| \leq |B| . |A| \leq \frac{n^2}{4}$

SUM TO n

$|B| . (n - |B|) \leq \frac{n^2}{4}$

MAX IF EQUAL

$$e(G) + e(G[B]) \leq \frac{n^2}{4}$$

$$\left\lfloor \frac{n^2}{4} \right\rfloor - t + e(G[B]) \leq \frac{n^2}{4}$$

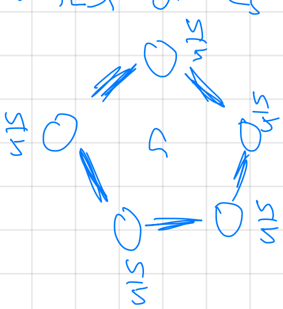
$$e(G[B]) \leq t$$

NOW WE REMOVE (AT MOST t) EDGES INSIDE B TO OBTAIN THE SUBGRAPH H THAT IS BIPARTITE (A, B IS THE BIPARTITION)

DISCUSSION

- IF $t = 0$, THERE IS G Δ-FREE WITH $e(G) = \left\lfloor \frac{n^2}{4} \right\rfloor$, ALREADY BIPARTITE BUT NOTHING TO REMOVE

- TEST ON C_5 BLOW-UP



$e(G) = 5 . \frac{n}{5} . \frac{n}{5} = \frac{n^2}{5} = \frac{n^2}{4} - t \Rightarrow t = \frac{n^2}{20}$

THIS IS A BUDGET $\frac{n^2}{20}$ TO MAKE G BIPARTITE

SUBGRAPH H WITH $e(H) \geq e(G) - \frac{n^2}{25}$

$$e(G) - \frac{n^2}{25} = e(G) - \frac{4}{5} t$$

WE MANAGED TO MAKE IT BIPARTITE WITH $\frac{4}{5} t$ INSTEAD OF t

- **FÜREDI UPDATED:**

LET G BE A TRIANGLE-FREE (K_3-FREE) GRAPH ON n VERTICES WITH

$$e(G) = \left\lfloor \frac{n^2}{4} \right\rfloor - t \quad \text{FOR SOME } t \geq 0$$

THEN THERE EXISTS A BIPARTITE SUBGRAPH $H \subseteq G$ S.T.

HU, LIDICKY, MARTINS, VOLEC, MORIN 2021

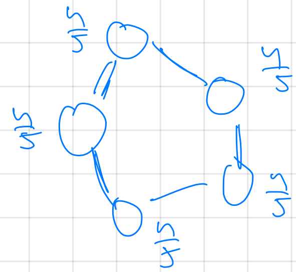
$\cancel{e(H) \geq e(G) - t}$ $\qquad$ $e(H) \geq e(G) - 0.8t$

• CONJECTURE (ERDŐS 1975)

LET G BE A K_3-FREE GRAPH ON n VERTICES THEN THERE EXISTS A BIPARTITE SUBGRAPH H WITH

$$e(H) \ge e(G) - \frac{n^2}{25}$$

IF TRUE, BEST POSSIBLE



1988 ERDŐS, FÜREDI, PACH, SPENCER $e(H) \ge e(G) - \frac{n^2}{18}$

2021: BALOGH, CLEMEN, LIDICKÝ $e(H) \ge e(G) - \frac{n^2}{23.5}$

STABILITY: "IF G CLOSE TO EXTREMAL GRAPH, THEN G LOOKS LIKE THE EXTREMAL CONSTRUCTION"

SOME PROBLEMS HAVE STABILITY, SOME DON'T

• $H = K_3$ MANTEL'S THEOREM

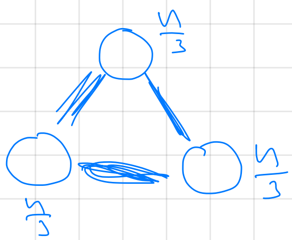
• $H = K_{r+1}$ (TURÁN THEOREM)

$$ex(n, K_{r+1}) = \left(1-\frac{1}{r}\right)\frac{n^2}{2} - O(1)$$

SMALL CONSTANT, FOR DIVISIBILITY

LOWER BOUND

$H = K_4$:



$H = K_{r+1}$